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AN EXTENSION OF A THEOREM OF HARTOGS

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Abstract. Hartogs proved that every function which is holomorphic on the boundary of the unit ball in C^n, $n>1$, can be extended to a function holomorphic on the ball itself. It is conjectured that a real k-dimensional $\mathscr{C}^\infty$ compact submanifold of C^n, $k>n$, is extendible over a manifold of real dimension $(k+1)$. This is known for hypersurfaces (i.e., $k=2n-1$) and submanifolds of real codimension 2. It is the purpose of this paper to prove this conjecture and to show that we actually get C-R extendibility.

1. **Introduction.** Let M^k be a real k-dimensional compact $\mathscr{C}^\infty$ manifold embedded in C^n, $k, n \geqq 2$. Hartogs proved that every function holomorphic in an open neighborhood of M^{2n-1} can be extended to a function holomorphic in some open subset of C^n. Bochner proved a similar theorem for functions which satisfy the induced Cauchy-Riemann equations on M^{2n-1}. It has been conjectured that any real k-dimensional compact $\mathscr{C}^\infty$ submanifold of C^n is extendible to a manifold of real dimension $(k+1)$ if $k>n$. This has been proved for real-analytic submanifolds of C^n in [3] and generic C-R submanifolds in [2]. It is the purpose of this paper to prove the conjecture with extendibility being replaced by C-R extendibility.

The early work for the higher codimensional study was done by Bishop [1], Wells [6] and Greenfield [2]. A recent article due to Nirenberg [4] led to the results in this paper.

2. **Definitions.** Let M^k be a real k-dimensional $\mathscr{C}^\infty$ manifold embedded in C^n, $k, n \geqq 2$. Suppose $T(M^k)$ is the tangent bundle to M^k, and J denotes the almost complex tensor $J: T(C^n) \to T(C^n)$, with $J^2=-I$. Then we define

$$H_p(M^k) = T_p(M^k) \cap JT_p(M^k),$$

the vector space of *holomorphic tangent vectors* to M^k at p. Then $H_p(M^k)$ is the maximal complex subspace of $T_p(C^n)$ which is contained in $T_p(M^k)$. It is well known that

$$\max(k-n, 0) \leqq \dim_{C} H_p(M^k) \leqq [k/2].$$

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There is another way of examining almost complex structures which we shall use. Let f denote the embedding of M^k into C^n, and let $J(f)$ be the complex Jacobian of f. If $q=\min(n, k)$, a point p in M^k is said to be an *exceptional point of order* l, $0 \leqq l \leqq [k/2] - \max(k-n, 0)$ if the complex rank of $J(f)|_p$ is equal to $q-l$.

A point p in M^k is *generic* if p is an exceptional point of order 0. The manifold M^k is *locally generic* at p if every point in some open neighborhood of p is generic, and is *locally C-R* at p if every point in some open neighborhood of p is an exceptional point of the same order.

Suppose M^k is locally C-R at p and $H_p(M^k)$ is nonempty. Then we define the *Levi form* at any x near p

$$L_x(M^k)\colon H_x(M^k) \to (Tx(M^k) \otimes C)/(H_x(M^k) \otimes C)$$

by $L_x(M^k)(t) = \pi_x\{[Y, \bar{Y}]_x\}$, where Y is a local section of the fiber bundle $H(M^k)$ (with fiber $H_x(M^k)$) such that $Y_x = t$, $[Y, \bar{Y}]_x$ is the Lie bracket evaluated at x, and

$$\pi_x\colon T_x(M^k) \otimes C \to (T_x(M^k) \otimes C)/(H_x(M^k) \otimes C)$$

is the projection.

Denote by $\mathcal{O}_{C^n} = \mathcal{O}$ the sheaf of germs of holomorphic functions on C^n. Let K be a compact subset of C^n and V an open subset of C^n containing K. We set

$$\mathcal{O}(K) = \operatorname*{ind\,lim}_{V \supset K} \mathcal{O}(V),$$

where $\mathcal{O}(V)$ is the Fréchet algebra of holomorphic functions on V. We say that K is *extendible* to a connected set $K' \supsetneqq K$ if the map $r\colon \mathcal{O}(K') \to \mathcal{O}(K)$ is onto.

Suppose $f \in \mathscr{C}^\infty(M^k)$. We say f is a *C-R function at* $p \in M^k$ if $\bar{X}f(y) = 0$, for y near p and X any section of $H(M^k)$. If M^k is locally C-R at p it suffices to verify the equality just for X in a local basis for $H(M^k)$ at p. We note that our manifold need not be globally C-R. Thus we may have points which are not locally C-R. But obviously, the set of such points is nowhere dense in M^k.

Definition 2.1. Let $f \in \mathscr{C}^\infty(M^k)$. Then f is a *C-R function on* M^k if f is a C-R function at each point of M^k. The C-R functions are denoted by CR (M^k).

We say that M^k is *C-R extendible* to a connected set $K = M^k \cup K'$, where $K' \neq \varnothing$, if for every $f \in \mathrm{CR}(M^k)$ there exists an $F\colon M^k \cup K' \to C$ continuous so that $F|_{M^k} = f$ and $F|_{K'} \in \mathcal{O}(K')$. We observe that C-R extendibility implies extendibility.

Let K be a compact subset of C^n. We shall call a point $x \in K$ a *holomorphic peak point* if there exists a function $f \in \mathcal{O}(K)$ such that, for any $y \in K - \{x\}$, we have $|f(y)| < |f(x)|$.

3. **Local equations and the Levi form.** Again let M^k be a real k-dimensional $\mathscr{C}^\infty$ manifold embedded in C^n, $k, n \geqq 2$. Suppose M^k is locally C-R at p, and p is an exceptional point of order l. If $k > n$ the local equations of M^k in a neighborhood of p are (after a suitable choice of coordinates)

$$
\begin{aligned}
&z_1 = x_1 + ih_1(x_1, \ldots, x_{2(n-l)-k}, w_1, \ldots, w_{k-n+l}) \\
&\vdots \\
&z_{2(n-l)-k} = x_{2(n-l)-k} + ih_{2(n-l)-k}(x_1, \ldots, x_{2(n-l)-k}, w_1, \ldots, w_{k-n+l}) \\
&z_{2(n-l)-k+1} = u_1 + iv_1 = w_1 \\
&\vdots \\
&z_{n-l} = u_{k-n+l} + iv_{k-n+l} = w_{k-n+l} \\
&z_{n-l+1} = g_1(x_1, \ldots, x_{2(n-l)-k}, w_1, \ldots, w_{k-n+l}) \\
&\vdots \\
&z_n = g_l(x_1, \ldots, x_{2(n-l)-k}, w_1, \ldots, w_{k-n+l}),
\end{aligned}
\tag{1}
$$

where $x_1, \ldots, x_{2(n-l)-k}, u_1, v_1, \ldots, u_{k-n+l}, v_{k-n+l}$ are local coordinates for M^k in a neighborhood of p vanishing at p, and $z_1, \ldots, z_n$ are coordinates for C^n vanishing at p. The real-valued functions $h_1, \ldots, h_{2(n-l)-k}$ as well as the complex-valued functions $g_1, \ldots, g_l$ vanish to order 2 at p. Because M^k is locally C-R at p, the functions $g_1, \ldots, g_l$ must be complex-analytic functions of $w_1, \ldots, w_{k-n+l}$ (see [3]).

Letting $g_j = g_j' + ig_j''$, $j = 1, \ldots, l$, we find from [5] that the Levi form vanishes at p if and only if the complex Hessians at p of each of the functions $h_1, \ldots, h_{2(n-l)-k}$, $g_1', g_1'', \ldots, g_l', g_l''$ with respect to the variables $w_1, \ldots, w_{k-n+l}$ all have zero eigenvalues.

Fix $x_1, \ldots, x_{2(n-l)-k}$ and expand each g_j in a Taylor series in $w_1, \ldots, w_{k-n+l}$,

$$g_j = \sum_\alpha a_{j,\alpha} w^\alpha,$$

where $w = (w_1, \ldots, w_{k-n+l})$ and $\alpha = (\alpha_1, \ldots, \alpha_{k-n+l})$. Replacing z_{n-l+j} by $z_{n-l+j} - \sum_\alpha a_{j,\alpha} w^\alpha$, we have that $z_{n-l+1} = 0, \ldots, z_n = 0$ in our new local equations. Thus the Levi form vanishes at p if and only if the complex Hessians at p of each of the functions $h_1, \ldots, h_{2(n-l)-k}$ are all zero matrices.

Suppose M^k is compact in C^n. It is shown in [5] that there exists an open set of holomorphic peak points on M^k which is nonempty. By the remarks before Definition 2.1, we can find a holomorphic peak point $p \in M^k$ such that p is an exceptional point of some order l, and M^k is locally C-R at p. Assume $p = 0$ and M^k near p is given by the equations in (1). Wells proves that through p we can put a hyperplane which intersects M^k at only the point p. If $z_j = x_j + iy_j$, $j = 1, \ldots, 2(n-l)-k, n-l+1, \ldots, n$, we can assume the hyperplane is defined by $y_1 = 0$ (the information about the g_j's in this section forces our arbitrary choice to $y_1, \ldots, y_{2(n-l)-k}$).

Let Q denote the 1-dimensional real subspace of $T_0(C^n)$ generated by $\partial/\partial y_1$. Set

$W = Q \oplus T_0(M^k)$ and let π be the projection from C^n to W. Under this projection the manifold M^k projects to a manifold with local equations

$$
(2)\qquad
\begin{aligned}
z_1 &= x_1 + ih_1(x_1, \ldots, x_{2(n-l)-k}, w_1, \ldots, w_{k-n+l}) \\
z_2 &= x_2 \\
&\vdots \\
z_{2(n-l)-k} &= x_{2(n-l)-k} \\
z_{2(n-l)-k+1} &= u_1 + iv_1 = w_1 \\
&\vdots \\
z_{n-l} &= u_{k-n+l} + iv_{k-n+l} = w_{k-n+l}.
\end{aligned}
$$

Wells shows that

$$
\frac{\partial^2 h_1}{\partial x_1^2}, \ldots, \frac{\partial^2 h_1}{\partial x_{2(n-l)-k}^2}, \frac{\partial^2 h_1}{\partial u_1^2}, \ldots, \frac{\partial^2 h_1}{\partial u_{k-n+l}^2}, \frac{\partial^2 h_1}{\partial v_1^2}, \ldots, \frac{\partial^2 h_1}{\partial v_{k-n+l}^2}
$$

are all >0 on some open neighborhood U of p in M^k. In particular

$$
\frac{\partial^2 h_1}{\partial w_1 \, \partial \bar{w}_1}, \ldots, \frac{\partial^2 h_1}{\partial w_{k-n+l} \, \partial \bar{w}_{k-n+l}}
$$

are positive on the set U. By diagonalizing, we find that the Hessian of h_1 with respect to $w_1, \ldots, w_{k-n+l}$ is positive definite.

4. **The main result.** Assume M^k is a real k-dimensional $\mathscr{C}^\infty$ manifold embedded in C^n, and M^k is locally C-R at $p \in M^k$. Suppose at least one of the following conditions is satisfied.

(I) There is a real hypersurface containing M^k whose Levi form restricted to $H(M^k)$ has at p at least one positive and one negative eigenvalue.

(II) There is a real hypersurface containing M^k whose Levi form restricted to $H(M^k)$ has at p all its eigenvalues of the same sign different from zero.

Then we have the following theorem due to Nirenberg [4].

THEOREM 4.1. *Let M^k be locally C-R at $p \in M$ and assume either* (I) *or* (II) *holds. Then M^k is locally C-R extendible to a manifold $\tilde{M}$ of real dimension one higher than that of M^k.*

We are now able to prove the main result.

THEOREM 4.2. *Let M^k be a real k-dimensional compact $\mathscr{C}^\infty$ manifold embedded in C^n, $k > n \geqq 2$. Then M^k is C-R extendible to a real* $(k+1)$*-dimensional submanifold of C^n.*

Proof. We showed in the previous section that there exists a point $p \in M^k$ such that:

(i) M^k is locally C-R at p,

(ii) M^k is given by the local equations (1) near p, and

(iii) the complex Hessian of the function h_1 with respect to the variables $w_1, \ldots, w_{k-n+l}$ has all positive eigenvalues at p.

Consider the real hypersurface containing M^k defined by the function $\rho = y_1 - h_1$. The Levi form of this hypersurface restricted to $H(M^k)$ is the negative of the complex Hessian of h_1 with respect to the variables $w_1, \ldots, w_{k-n+l}$. Then this hypersurface satisfies condition (II) at the point p, and we apply Theorem 4.1. Q.E.D.

THEOREM 4.3. *Let M^k be a real k-dimensional compact $\mathscr{C}^\infty$ manifold embedded in $\mathbf{C}^n$, $k > n \geqq 2$. Then M^k is extendible to a real $(k+1)$-dimensional submanifold of $\mathbf{C}^n$.*

REMARK 1. The manifold $\tilde{M}$ of Theorem 1 can be taken to have $\mathscr{C}^q$ structure, $1 \leqq q < \infty$.

REMARK 2. If $k \leqq n$, then there are examples of totally real submanifolds which are always holomorphically convex. Thus, from the standpoint of dimension, Theorems 4.2 and 4.3 are the best possible.

REFERENCES

1. E. Bishop, *Differentiable manifolds in complex Euclidean space*, Duke Math. J. **32** (1965), 1–21. MR **34** #369.

2. S. J. Greenfield, *Cauchy-Riemann equations in several variables*, Ann. Scuola Norm. Sup. Pisa (3) **22** (1968), 275–314. MR **38** #6097.

3. L. R. Hunt, *Real-analytic submanifolds of complex manifolds*, Proc. Amer. Math. Soc. **29** (1971), 69–74.

4. R. Nirenberg, *On the H. Lewy extension phenomenon* (to appear).

5. R. O. Wells, Jr., *Concerning the envelope of holomorphy of a compact differentiable submanifold of a complex manifold*, Ann. Scuola Norm. Sup. Pisa (3) **23** (1969), 347–361. MR **39** #7141.

6. ———, *Holomorphic hulls and holomorphic convexity of differentiable submanifolds*, Trans. Amer. Math. Soc. **132** (1968), 245–262. MR **36** #5392.

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